CHAPTER SEVEN
ROOT FINDING OF EQUATIONS

• Engineers are often required to solve complicated equations.

• Or it is required to obtain the *Roots of Equations.*

• There are different types of mathematical equations:

a) Algebraic equations such as:

$$f(x) = a_0 + a_1 x + \dots + a_n x^n$$

or $$f(x) = 1-2.37 x + 7.5 x^2$$

or $$f(x) = 5x^2 - x^3 + 7x^6$$

b) Non-Algebraic equation (Transcendental) such as:

$f(x) = \sin x$

or $$f(x) = \ln x^2-1$$

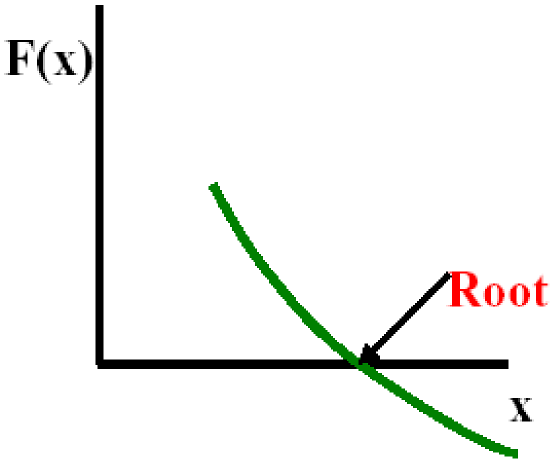
Required:

1 - The determination of the real roots of the equations.

2 - Develop a spread sheet solution to obtain the roots of the equations.

Bracketing Methods

These techniques exploit the fact that a function typically changes sign in the vicinity of a root.

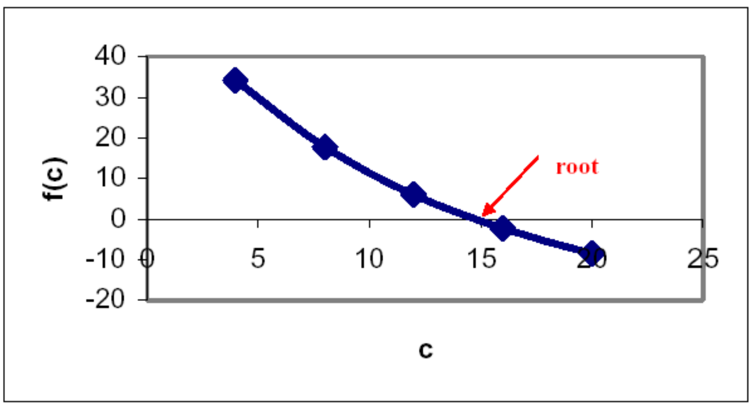


1 - Graphical Methods:

• Use the graphical approach to determine the roots of the equation:

c	4	8	12	16	20
F(c)	34.115	17.653	6.067	-2.269	-8.401

From graph: c **approximately = 15**



2 – THE BISECTION METHOD

It is based on the observation:

• If f(x) is a real function and continuous in the interval between xl and xu

• and if f(x1) * f(xu) < 0

• Then there should be at least one real root between xl and xu.

ALGORITHM FOR BISECTION :

Step 1: Choose lower xl and upper xu guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that f(xl)*f(xu) < 0.

Step 2: An estimate of the foot xr, is determined by

Xr = (Xl+ Xu)/2

Step 3: Make the following evaluations to determine in which subinterval the root lies:

(a) If $f(x_l)\, f(x_r) < 0$, the root lies in the lower subinterval.

Therefore, set $x_u = x_r$, and return to step 2.

(b) If $f(x_l) f(x_r) > 0$, the root lies in the upper subinterval.

Therefore, set $x_l = x_r$ and return to step 2.

(c) If $f(x_l) f(x_r) = 0$, the root equals x_r.

Terminate the computation.

<u>Use the Bisection method to obtain the root of the equation</u>

$$f(c) = \frac{667.38}{c}(1 - e^{-0.146843c}) - 40$$

Step 1: choose (guess) two values of " c " that gives values of f(c) with different signs.

take cl = 12 and cu = 16

f(cl) * f (cu) = 6.067 * (- 2.269) < 0.0

Step 2: *An estimate of the root cr is given by*

cr = (cl + cu)/2 = (12+16)/2 =14

Step 3: *check where does the root lie?*

f (12) f (14) = 6.067 * 1.569 = 9.514 > 0.0

The root must be located between

c = 14 and c = 16

Set cl = 14 and return to step 2

Step 2 :

$$Cr = \frac{14 + 16}{2} = 15$$

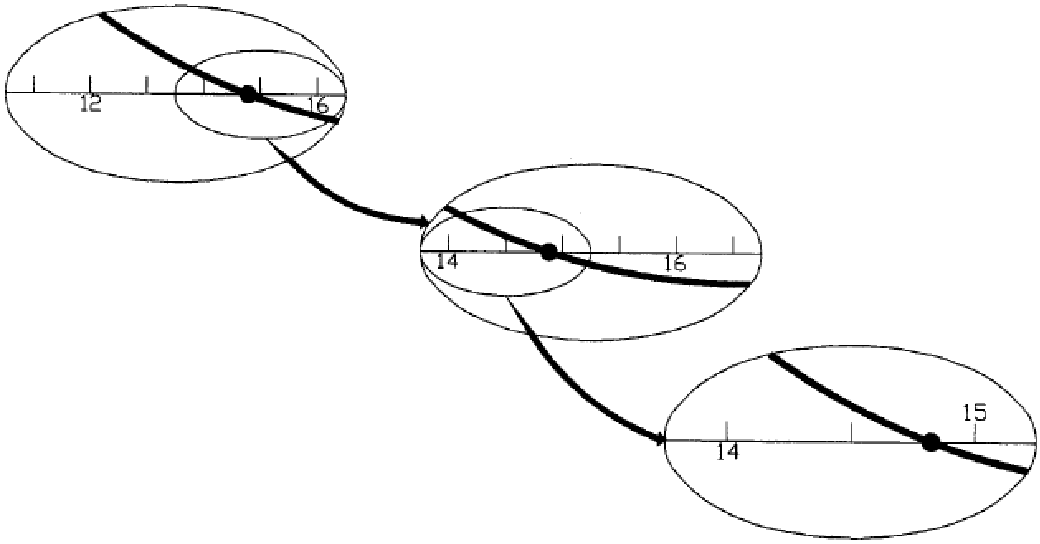
F (15) f (14) = 1.569 (- 0.425) = -0.666 < 0.0

The root is between 14 and 15

Step 2:

$$Cr = \frac{14 + 15}{2} = \mathbf{14.5}$$

Solving the problem in a tabular form:



Iteration no.	C_l	C_u	$F(C_l)$	$F(C_u)$	C_r	$F(C_r)$	Check	Error %
0	12	16	6.06202	-2.27149	14	1.56506	9.48728	
1	14	16	1.56506	-2.27149	15	-0.428	-0.66984	6.66667
2	14	15	1.56506	-0.428	14.5	0.54891	0.85907	-3.44828
3	14.5	15	0.54891	-0.428	14.75	0.05567	0.03056	1.69491

Termination Criteria and Error Estimates:

• We can stop the procedure when the error in calculating the root drops below a certain value

$$\varepsilon_t = \text{true error } \%$$

$$= \frac{\text{true root - approximation}}{\text{true root}} * 100$$

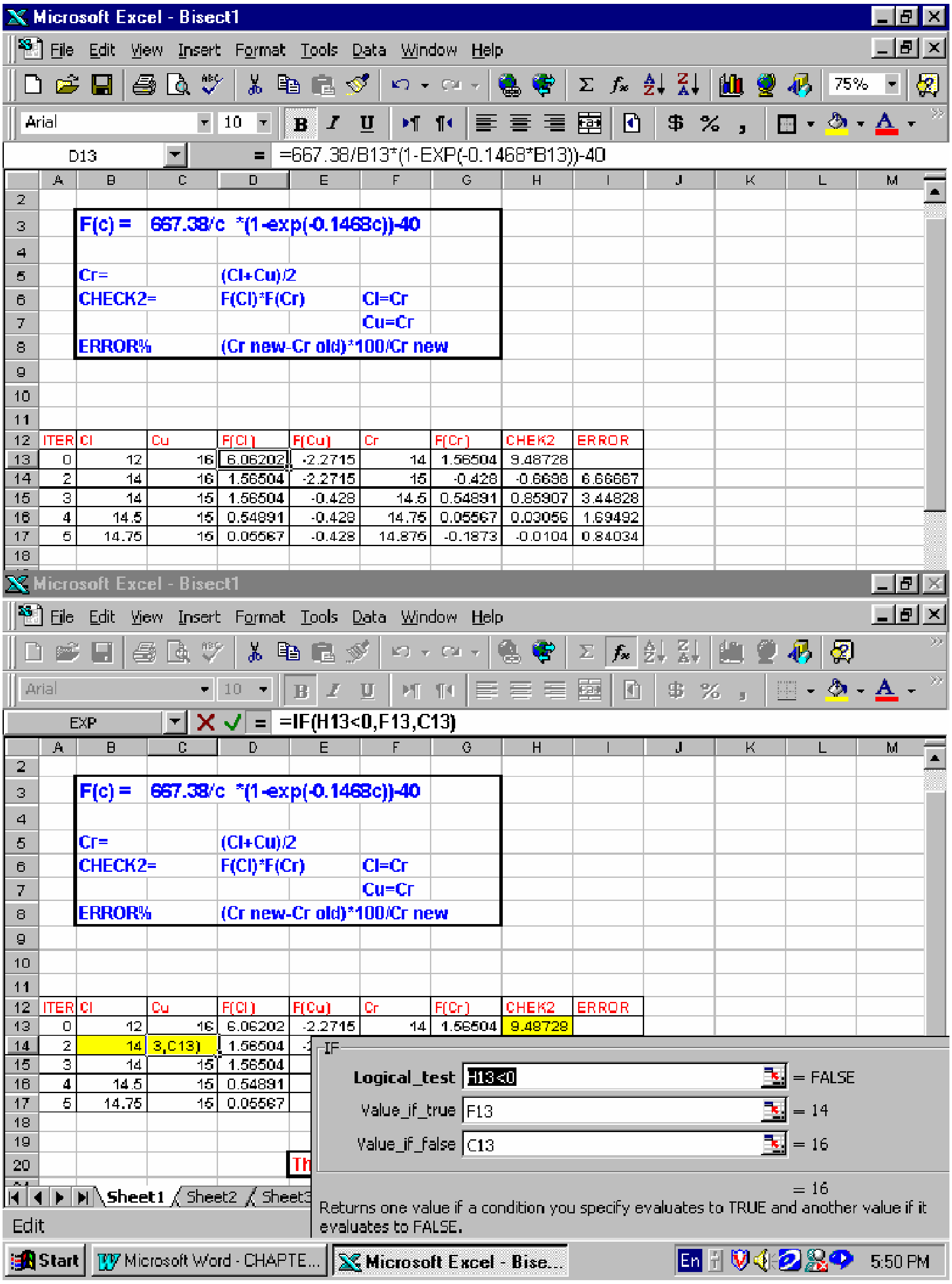
but we do not know the true root

• Use an approximate relation error

$$\varepsilon_t = \text{approximate error } \%$$

$$|\varepsilon_t| = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| 100 \ \%$$

Using Excel to solve the Bisection method:



Incremental searches and Determining Initial Guesses:

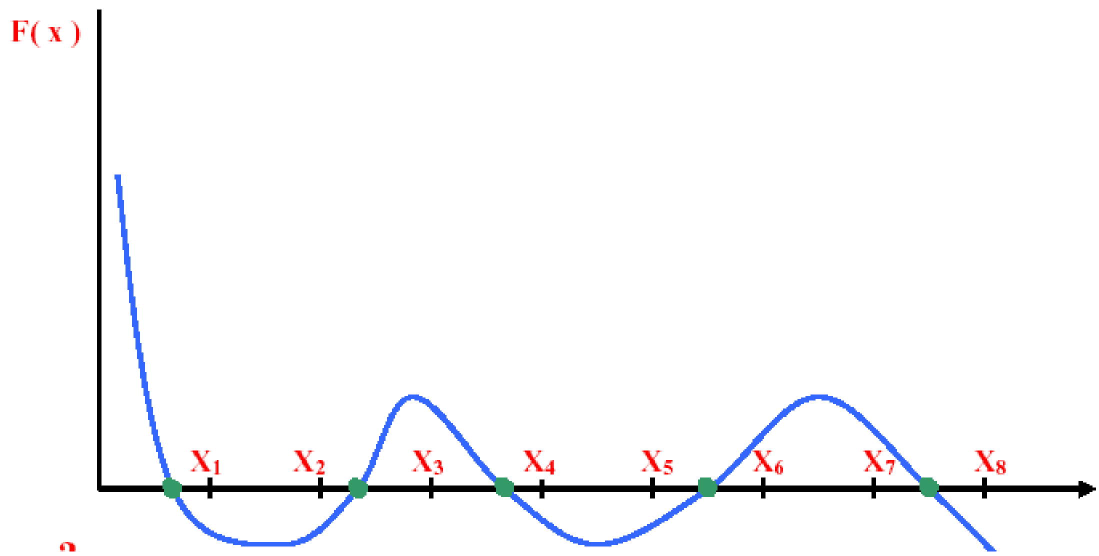
To obtain all possible roots we use an *Incremental search:*

• Start at one end of the region of interest

• Make function evaluations at small increments across the region.

• When the function changes sign, it is assumed that a root falls within the increment.

• The x values at the beginning and the end of the increment can serve as the initial guesses of xl and xu.

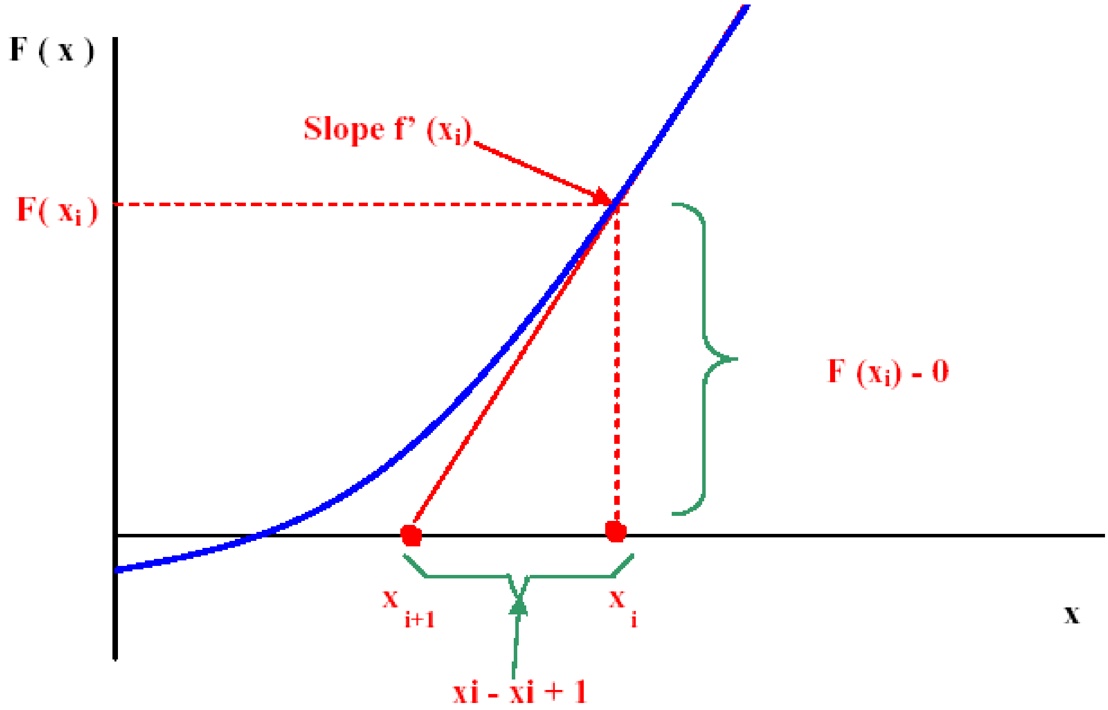


Open Methods :

• Based on formulas that require a single starting value x or two starting values that do necessarily bracket the root.

. These methods sometimes diverge or move from the true root.

Newton – Raphson Method:



$$f'(x_i) = \frac{f(x_i) - 0.0}{x_i - x_{i+1}}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Newton – Raphson Formula

Example:

Use the Newton – Raphson Method to estimate the root of

$f(x) = e^{-x} - x.$

Use initial guess of $x_0 = 0$.

Solution:

$\therefore \quad f'(x) = -e^{-x} - 1$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

$$x_{i+1} = x_i - (e^{-x_i} - x_i) / (-e^{-x_i} - 1)$$

Starting with $X_0 = 0.0$

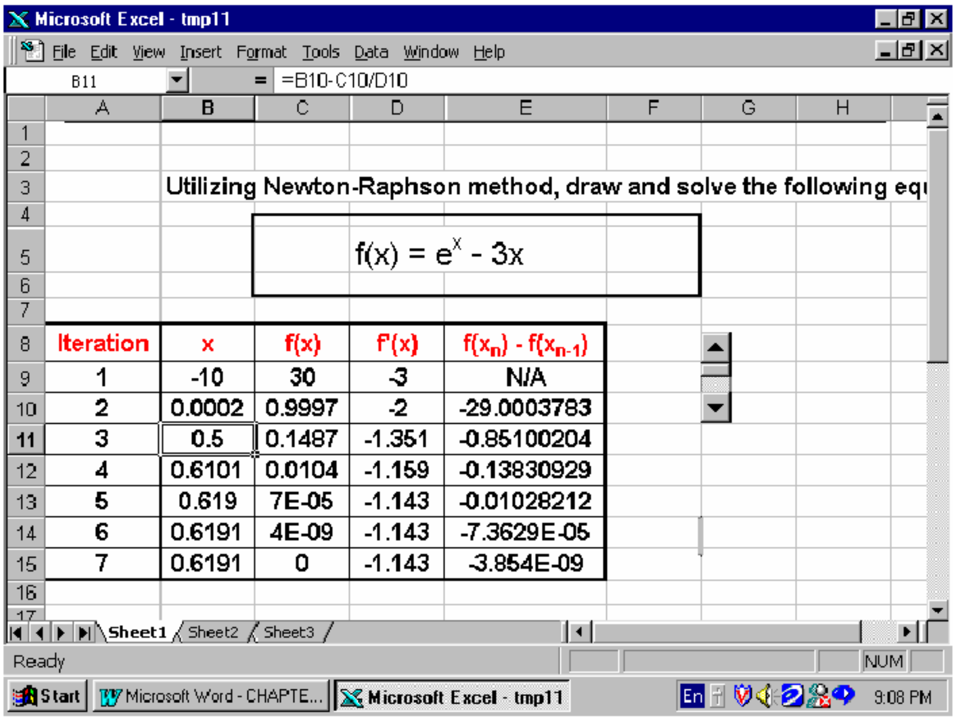
No. of iteration	root
i	x i
0	0
1	0.5000000
2	0.566311003
3	0.567143165
4	0.567143290

Termination Criteria

when to stop iteration ?

$\varepsilon_a = $ ABS $[(x_i - x_{i-1}) / x_I]$ * 100 % < Subscribed value

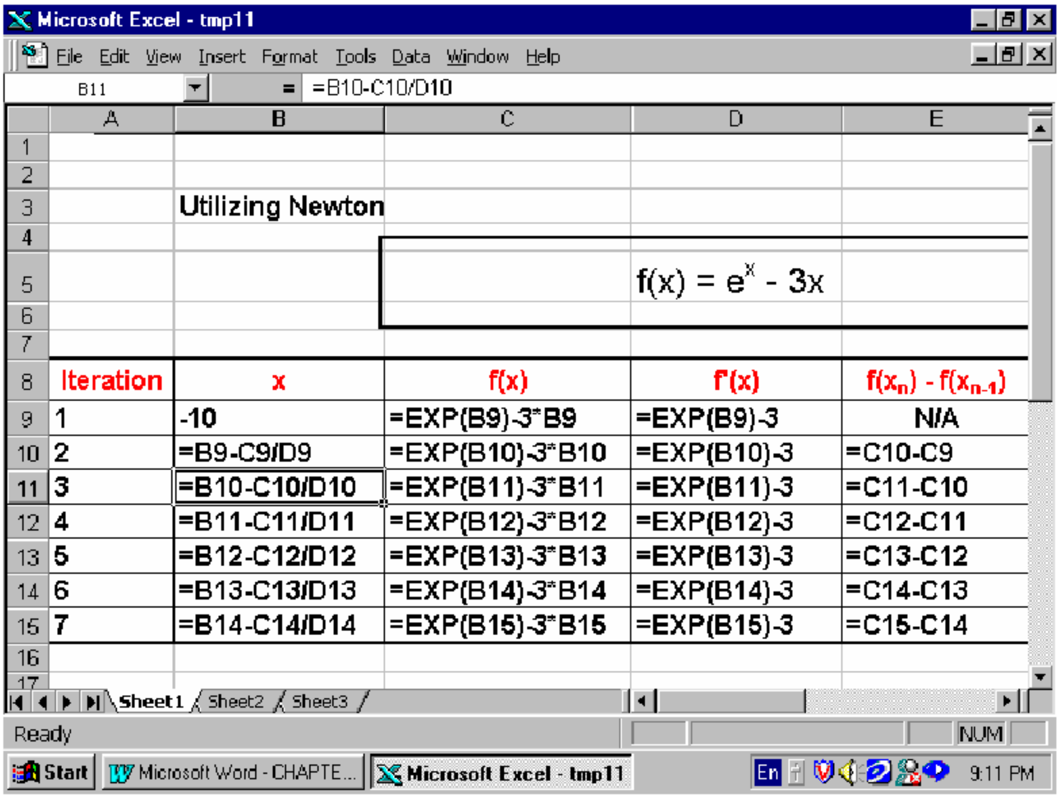
Using Excel to solve Newton-Raphson Problems



Utilizing Newton-Raphson method, draw and solve the following equ

$f(x) = e^x - 3x$

Iteration	x	f(x)	f'(x)	$f(x_n) - f(x_{n-1})$
1	-10	30	-3	N/A
2	0.0002	0.9997	-2	-29.0003783
3	0.5	0.1487	-1.351	-0.85100204
4	0.6101	0.0104	-1.159	-0.13830929
5	0.619	7E-05	-1.143	-0.01028212
6	0.6191	4E-09	-1.143	-7.3629E-05
7	0.6191	0	-1.143	-3.854E-09



Utilizing Newton

$f(x) = e^x - 3x$

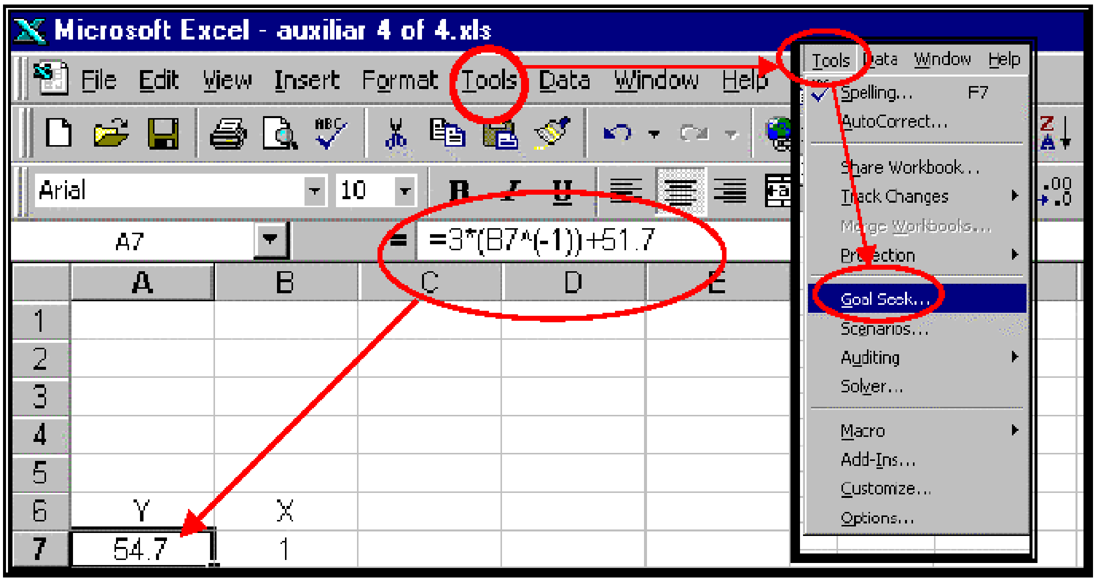
Iteration	x	f(x)	f'(x)	$f(x_n) - f(x_{n-1})$
1	-10	=EXP(B9)-3*B9	=EXP(B9)-3	N/A
2	=B9-C9/D9	=EXP(B10)-3*B10	=EXP(B10)-3	=C10-C9
3	=B10-C10/D10	=EXP(B11)-3*B11	=EXP(B11)-3	=C11-C10
4	=B11-C11/D11	=EXP(B12)-3*B12	=EXP(B12)-3	=C12-C11
5	=B12-C12/D12	=EXP(B13)-3*B13	=EXP(B13)-3	=C13-C12
6	=B13-C13/D13	=EXP(B14)-3*B14	=EXP(B14)-3	=C14-C13
7	=B14-C14/D14	=EXP(B15)-3*B15	=EXP(B15)-3	=C15-C14

Solving Equations in Excel Using Goal Seek:

When you know the desired result of a single formula but not the input value the formula needs to determine the result, you can use the "**Goal Seek**" feature. When "***Goal Seeking***", MS Excel varies the value in one specific cell until a formula that's dependent on that cell returns the result you want.

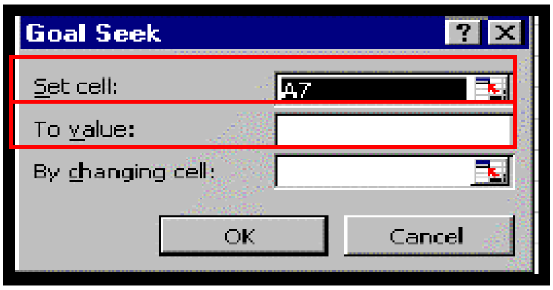
For instance, suppose that you have the following equation:

$Y = 3 \times X^{-1} + 11.7$, and you have to determine the X value for a Y value equals to 51.7. Please, return to your worksheet and type the equation in cell A7. Type "***1***" in cell B7 and the click on "**Goal Seek**" under the "**Tools**" menu:



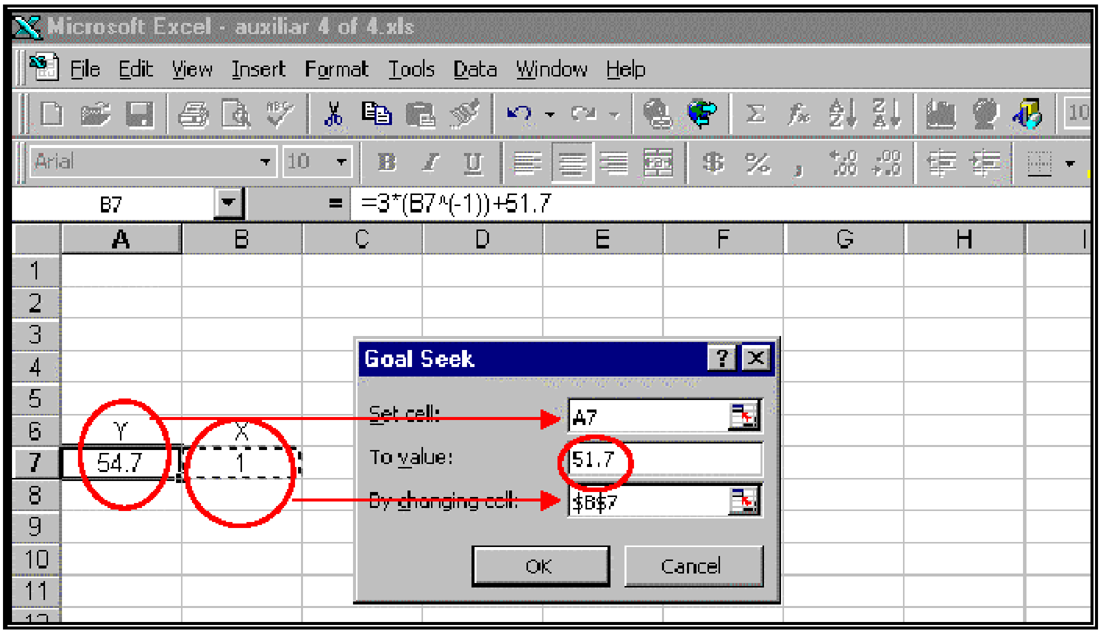
Selecting the Goal Seek" Feature

A “**Goal Seek**” window becomes visible. Type the cell containing the value in the “**Set cell**” box, and the desired value in the “**To value**” box.



The “Goal Seek” Window

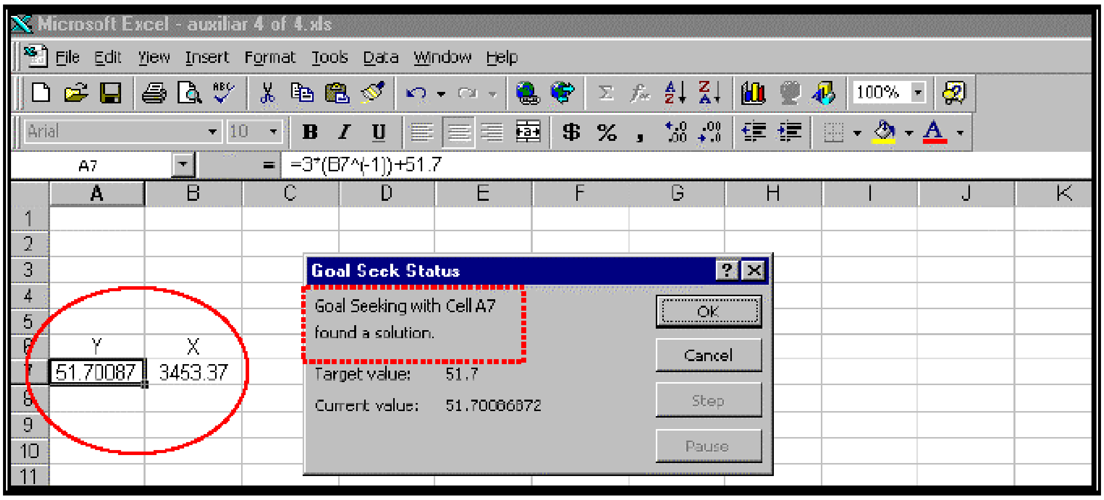
In the “**By changing cell**” box type the cell containing the variable X:



Entering Data into the “Goal Seek” Window

After clicking on the “**OK**” button, a “**Goal Seek Status**” becomes visible informing if MS Excel could or could not find the solution.

Notice that both the result and the X value are displayed in the worksheet:



Result of Using the "Goal Seek" Feature