# CHAPTER SEVEN ROOT FINDING OF EQUATIONS

- Engineers are often required to solve complicated equations.
- Or it is required to obtain the *Roots of Equations*.
- There are different types of mathematical equations:

a) Algebraic equations such as:

$$f(x) = a_0 + a_1 x + ... + a_n x^n$$

or 
$$f(x) = 1-2.37 x + 7.5 x^2$$

or 
$$f(x) = 5x^2 - x^3 + 7x^6$$

b) Non-Algebraic equation (Transcendental) such as:

 $f(x) = \sin x$ or  $f(x) = \ln x^2 - 1$ 

Required:

1 - The determination of the real roots of the equations.

2 - Develop a spread sheet solution to obtain the roots of the equations.

#### Bracketing Methods

These techniques exploit the fact that a function typically changes sign in the vicinity of a root.



1 - Graphical Methods:

• Use the graphical approach to determine the roots of the equation:

c	4	8	12	16	20
F(c)	34.115	17.653	6.067	-2.269	-8.401

From graph: **c** approximately = 15



#### **2 – THE BISECTION METHOD**

It is based on the observation:

- If f(x) is a real function and continuous in the interval between xl and xu
- and if f(x1) \* f(xu) < 0
- Then there should be at least one real root between xl and xu.

### **ALGORITHM FOR BISECTION :**

**Step 1:** Choose lower xl and upper xu guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that f(xl)\*f(xu) < 0.

Step 2: An estimate of the foot xr, is determined by

Xr = (Xl + Xu)/2

**Step 3:** Make the following evaluations to determine in which subinterval the root lies:

(a) If  $f(x_{l}) f(x_{r}) < 0$ , the root lies in the lower subinterval.

Therefore, set  $X_u = X_r$ , and return to step 2.

(b) If  $f(x_{\nu}) > 0$ , the root lies in the upper subinterval.

Therefore, set  $x_1 = x_r$  and return to step 2.

(c) If  $f(X_v)f(X_v) = 0$ , the root equals  $X_v$ .

Terminate the computation.

Use the Bisection method to obtain the root of the equation

$$f(c) = \frac{667.38}{c} (1 - e^{-0.146843c}) - 40$$

**Step 1:** choose (guess) two values of "c " that gives values of f(c) with different signs.

take cl = 12 and cu = 16 f(cl) \* f(cu) = 6.067 \* (-2.269) < 0.0 *Step 2:* An estimate of the root cr is given by cr = (cl + cu)/2 = (12+16)/2 = 14*Step 3:* check where does the root lie?

## f(12) f(14) = 6.067 \* 1.569 = 9.514 > 0.0

The root must be located between c = 14 and c = 16Set cl = 14 and return to step 2 Step 2 :

$$Cr = \frac{14 + 16}{2} = 15$$

F(15) f(14) = 1.569 (-0.425) = -0.666 < 0.0

The root is between 14 and 15

Step 2:

$$Cr = \frac{14 + 15}{2} = 14.5$$

Solving the problem in a tabular form:



Iteration	C	<b>C</b> <sub>u</sub>	F(C I)	F(C <sub>u</sub> )	C r	F(C <sub>r</sub> )	Check	Error %
no.								
0	12	16	6.06202	-2.27149	14	1.56506	9.48728	
1	14	16	1.56506	-2.27149	15	-0.428	-0.66984	6.66667
2	14	15	1.56506	-0.428	14.5	0.54891	0.85907	-3.44828
3	14.5	15	0.54891	-0.428	14.75	0.05567	0.03056	1.69491

Termination Criteria and Error Estimates:

• We can stop the procedure when the error in calculating the root drops below a certain value

$$\varepsilon_{\rm t}$$
 = true error %

but we do not know the true root

• Use an approximate relation error

$$\boldsymbol{\varepsilon}_{t} = \operatorname{approximate error} \%$$
$$\left|\boldsymbol{\varepsilon}_{t}\right| = \left|\frac{x_{r}^{\text{new}} - x_{r}^{\text{old}}}{x_{r}^{\text{new}}}\right| 100 \%$$

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16	4	14.5	15	0.54891	-0.428	14.75	0.0556	7 0.03056	1.69492				
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Using Excel to solve the Bisection method:

#### **Incremental searches and Determining Initial Guesses:**

To obtain all possible roots we use an Incremental search:

- Start at one end of the region of interest
- Make function evaluations at small increments across the region.
- When the function changes sign, it is assumed that a root falls within the increment.

• The x values at the beginning and the end of the increment can serve as the initial guesses of xl and xu.



#### **Open Methods :**

• Based on formulas that require a single starting value x or two starting values that do necessarily bracket the root.

. These methods sometimes diverge or move from the true root.



#### Example:

Use the Newton - Raphson Method to estimate the root of

 $\mathbf{f}(\mathbf{x}) = \mathbf{e}^{-\mathbf{x}} - \mathbf{x}.$ 

Use initial guess of  $\mathbf{x}_0 = \mathbf{0}$ .

#### **Solution:**

$$\therefore$$
 f'(x) = -e<sup>-x</sup> -1

$$\begin{array}{rcl} \mathbf{x}_{i+1} &= \mathbf{x}_i &- & \underline{\mathbf{f}(\mathbf{x}_i)} \\ & & \mathbf{f}'(\mathbf{x}_i) \end{array}$$

$$\mathbf{x}_{i+1} = \mathbf{x}_i - (\mathbf{e}^{-\mathbf{x}i} - \mathbf{x}_i) / (-\mathbf{e}^{-\mathbf{x}i} - 1)$$

Starting with  $X_0 = 0.0$ 

No. of iteration	root
i	x i
0	0
1	0.5000000
2	0.566311003
3	0.567143165
4	0.567143290

Termination Criteria

when to stop iteration?

 $\varepsilon_a = ABS [(x_i - x_{i-1}) / x_I] * 100 \% < Subscribed value$ 

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11	3	0.5	0.1487	-1.351	-0.85100204				
12	4	0.6101	0.0104	-1.159	-0.13830929				
13	5	0.619	7E-05	-1.143	-0.01028212				
14	6	0.6191	4E-09	-1.143	-7.3629E-05				
15	7	0.6191	0	-1.143	-3.854E-09				
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# Using Excel to solve Newton-Raphson Problems

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#### Solving Equations in Excel Using Goal Seek:

When you know the desired result of a single formula but not the input value the formula needs to determine the result, you can use the "Goal Seek" feature. When "Goal Seeking", MS Excel varies the value in one specific cell until a formula that's dependent on that cell returns the result you want.

For instance, suppose that you have the following equation:

 $Y = 3 \times X^{-1} + 11.7$ , and you have to determine the X value for a Y value equals to 51.7. Please, return to your worksheet and type the equation in cell A7. Type "*1*" in cell B7 and the click on "Goal Seek" under the "Tools" menu:



Selecting the Goal Seek" Feature

A "Goal Seek" window becomes visible. Type the cell containing the value in the "Set cell" box, and the desired value in the "To value" box.



The "Goal Seek" Window

In the "**By changing cell**" box type the cell containing the variable

X:

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Entering Data into the "Goal Seek" Window

After clicking on the "**OK**" button, a "**Goal Seek Status**" becomes visible informing if MS Excel could or could not find the solution.

Notice that both the result and the X value are displayed in the worksheet:

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**Result of Using the "Goal Seek" Feature**